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INTERACTION OF SHEAR FLOWS OF AN IDEAL INCOMPRESSIBLE FLUID IN A CHANNEL

A. A. Chesnokov

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The problem of the decay of an arbitrary discontinuity for the equations describing plane–parallel shear flows of an ideal fluid in a narrow channel is considered. The class of particular solutions corresponding to fluid flows with piecewise constant vorticity is studied. In this class, the existence of self-similar solutions describing all possible unsteady wave configurations resulting from the nonlinear interaction of the specified shear flows is established.

Key words: *shear flows, long waves, decay of an arbitrary discontinuity, hyperbolicity.*

Introduction. Many mathematical models describing the propagation of long-wave perturbations in shear (vortex) fluid flows reduce to nonlinear integrodifferential equations. A qualitative analysis of various models of long-wave theory was performed in [1] using the generalization of the hyperbolicity concept developed by Teshukov and the method of characteristics for systems of equations with operator coefficients [2]. The results of these studies show both differences and similarities between these models and hyperbolic differential systems (in particular, the presence of a continuous range of characteristic velocities). The evolution of solutions of generalized hyperbolic nonlinear integrodifferential equations can lead to the occurrence of strong discontinuities, which makes it necessary to correctly formulate the equations of motion in the form of conservation laws and to analyze the problem of the decay of an arbitrary discontinuity (Riemann problem).

In the present paper, the problem of the decay of an arbitrary discontinuity is considered for nonlinear equations describing shear flows with piecewise constant vorticity in a narrow channel. In this class, a self-similar solution is obtained and studied. In the region of shear flow interaction, the fluid flow is shown to have a substantially two-dimensional unsteady nature. This is manifested in the formation of jet flow along the interface between the flows, which is directed to the upper or lower boundary of the channel, depending on the vorticity ratio. A similar formulation was studied by Teshukov for a free-boundary model [3] with certain constraints imposed on the initial data to satisfy the conditions of strong nonlinearity of the characteristics. In the present paper, a solution of the shear flow interaction problem is constructed without constraints on their vorticities and the corresponding wave configurations that include a simple wave or a simple wave and a shock are analyzed. In the case of interaction of flows with arbitrary monotonic (in depth) velocity profiles, a discretization of the integrodifferential equations is proposed and differential conservation laws are derived.

1. Formulation of the Problem. The equations of plane–parallel motion of an ideal incompressible fluid in a channel are written as

$$u_t + uu_x + vu_y + \rho^{-1}p_x = 0, \qquad \varepsilon^2(v_t + uv_x + vv_y) + \rho^{-1}p_y = -g,$$
$$u_x + v_y = 0, \qquad v(t,x,0) = 0, \qquad v(t,x,h_0) = 0.$$

Here the dimensionless variables t, x, y, u, v, and p are the time, Cartesian coordinates, and velocity and pressure components; the constants ρ, g, and h_0 are the density, acceleration of gravity, and channel depth (without loss of generality, these constants can be set equal to unity); ε is the ratio of the vertical scale to the horizontal one,

Lavrent'ev Institute of Hydrodynamics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090; chesnokov@hydro.nsc.ru. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 47, No. 6, pp. 34–47, November–December, 2006. Original article submitted December 6, 2005.

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which is considered small. In the approximate theory, terms of order ε^2 can be ignored, resulting in a hydrostatic pressure distribution with depth: $p = g\rho(h_0 - y) + \rho p^*(t, x)$ (ρp^* is the dimensionless pressure on the upper cover of the channel). As a result, we obtain the following integrodifferential model:

$$u_t + uu_x + vu_y + p^*_x = 0, \qquad v = -\int_0^y u_x(t, x, y')\, dy', \qquad \int_0^{h_0} u_x(t, x, y')\, dy' = 0. \tag{1}$$

The last equation of system (1) implies that the fluid discharge in the channel

$$Q(t) = \int_0^{h_0} u(t, x, y)\, dy$$

does not depend on the variable x. Moreover, in the absence of sources, the discharge $Q(t)$ can be considered equal to zero since Eqs. (1) admit the transformation to the noninertial coordinate system

$$X = x - f(t), \qquad U = u - f'(t), \qquad P^* = p^* + xf''(t) \tag{2}$$

with an arbitrary function $f(t)$.

From (1) it follows that the functions $u_0(t, x)$ and $u_1(t, x)$ (the velocities on the upper and lower boundaries of the channel) satisfy the equations

$$u_{0t} + u_0 u_{0x} + p^*_x = 0, \qquad u_{1t} + u_1 u_{1x} + p^*_x = 0. \tag{3}$$

Using any of Eqs. (3), it is possible to eliminate the pressure p^* from system (1) and to reduce the integrodifferential model to the evolutionary form. Then, for the relative velocity $w(t, x, y) = u(t, x, y) - u_1(t, x)$, we obtain the equation

$$w_t + (w^2/2 + wu_1)_x + vw_y = 0, \tag{4}$$

where

$$v = -yu_{1x} - \int_0^y w_x(t, x, y')\, dy', \qquad u_1 = h_0^{-1}\Big(Q(t) - \int_0^{h_0} w(t, x, y)\, dy\Big).$$

To solve Eq. (4), it is necessary to specify the fluid discharge in the channel $Q(t)$ and the initial distribution of the relative velocity $w(0, x, y)$ such that $w(0, x, h_0) = 0$. Next, we assume that the discharge $Q = 0$ [by virtue of transformation (2), this does not limit the generality of the approach]. The initial data for Eq. (4) are specified as

$$w\Big|_{t=0} = \begin{cases} u^r(y) - u^r(h_0), & x > 0, \\ u^l(y) - u^l(h_0), & x < 0, \end{cases} \tag{5}$$

where $u^r(y)$ and $u^l(y)$ are arbitrary specified functions that satisfy the condition

$$Q = \int_0^{h_0} u^r(y)\, dy = \int_0^{h_0} u^l(y)\, dy = 0. \tag{6}$$

The initial data (5) are a generalization of the classical formulation of the Riemann problem for the integrodifferential equation (4). The solution of problem (4), (5) describes the unsteady wave configurations resulting from the interaction of the specified shear flows filling the channel.

In some cases, to analyze the characteristic properties and derive the conservation laws for long-wave models, it is expedient to use mixed Euler–Lagrangian variables [4]. In Eqs. (1), transformation to these coordinates is performed by changing the variable $y = \Phi(t, x, \lambda)$ [$\Phi(t, x, \lambda)$ is a solution of the Cauchy problem]:

$$\Phi_t + u(t, x, \Phi)\Phi_x = v(t, x, \Phi), \qquad \Phi(0, x, \lambda) = \Phi_0(x, \lambda) \qquad (0 \le \lambda \le 1).$$

The change of variables is invertible provided that $\Phi_\lambda \ne 0$. For the functions $\Phi_\lambda = u(t, x, \lambda)$ and $\Phi_\lambda = H(t, x, \lambda)$, we have the following system of differential equations with an additional integral condition:

$$(u - u_1)_t + ((u^2 - u_1^2)/2)_x = 0, \qquad H_t + (uH)_x = 0, \qquad \int_0^1 H\, d\lambda = h_0 \qquad (u_1 = u(t, x, 1)). \tag{7}$$

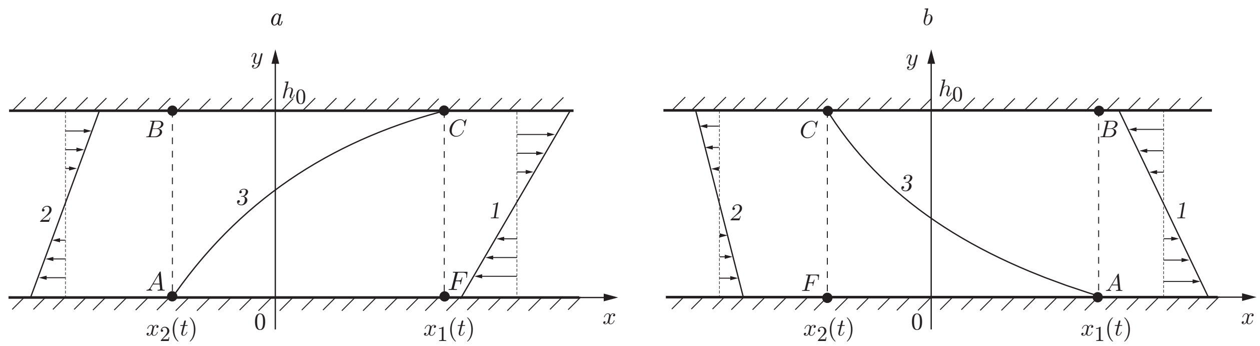

Fig. 1. Two-layer fluid flow with positive (a) and negative (b) piecewise constant vorticities: 1) flow velocity profile with vorticity ω_1; 2) the same with vorticity ω_2; 3) interface between the flows with vorticities ω_1 and ω_2.

In [5], system (7) is reduced to the evolutionary form

$$w_t + (w^2/2 + wu_1)_x = 0, \qquad H_t + (uH)_x = 0,$$
$$u_1 = -\Big(\int_0^1 H\,d\lambda\Big)^{-1} \int_0^1 wH\,d\lambda, \qquad u = w + u_1 \tag{8}$$

and studied using the generalization of the concepts of characteristics and hyperbolicity for systems with operator coefficients [1]. For flows with a monotonic-in-depth velocity profile ($u_\lambda \neq 0$), the characteristic equation

$$\chi(k) = -\frac{1}{\omega_1(u_1 - k)} + \frac{1}{\omega_0(u_0 - k)} + \int_0^1 \Big(\frac{1}{\omega}\Big)_\lambda \frac{d\lambda}{u - k} = 0 \tag{9}$$

was obtained, the existence of a continuous characteristic spectrum $k(t,x) = u(t,x,\lambda)$ was established, and the hyperbolicity conditions

$$\Delta \arg\,(\chi^+/\chi^-) = 0, \qquad \chi^\pm \neq 0$$

were formulated. Here $\omega = u_\lambda/H$; the complex functions $\chi^\pm(u(\lambda))$ are the limiting values of $\chi(k)$ from the upper and lower half-planes on the segment $[u_0, u_1]$; the increment of the argument of the function χ^+/χ^- is calculated for λ varying from 0 to 1; the subscripts 0 and 1 indicate that the functions are taken for $\lambda = 0$ and 1, respectively.

To determine the discontinuous solutions of the model of shear fluid flow in a channel, we use the system of conservation laws (7). The first equation is the local conservation law for the relative momentum, the second equation is the local mass conservation law, and the last equation is the closing one. The first two equations of system (7) coincide with the conservation laws proposed in [6, 7] to describe the discontinuous solutions of vortex shallow-water equations that model the propagation of long-wave perturbations in a free-boundary fluid layer. The conditions at the discontinuity corresponding to the conservation laws (7) are written as

$$[(u - D)^2 - (u_1 - D)^2] = 0, \qquad [(u - D)H] = 0, \qquad [Q] = 0. \tag{10}$$

Equations (10) imply the relation $[\omega] = [u_\lambda/H] = 0$ (the vorticity is conserved in passing through the discontinuity).

2. Equation of Two-Layer Flow with Piecewise Constant Vorticity. We consider the problem (1) with initial data of the form (5), in which

$$u^r(y) = \omega_1 y + u_0^r, \qquad u^l(y) = \omega_2 y + u_0^l$$

(ω_i, u_0^r, and u_0^l are constants). Using formula (6), we determine the velocities on the lower and upper covers of the channel:

$$u_0^l = -\omega_2 h_0/2, \qquad u_0^r = -\omega_1 h_0/2, \qquad u_1^l = \omega_2 h_0/2, \qquad u_1^r = \omega_1 h_0/2. \tag{11}$$

In the region of flow interaction (the region $ABCF$ in Fig. 1), the solution of Eqs. (1) is written as

$$u(t,x,y) = \begin{cases} \Omega_1 y + u_0(t,x), & 0 \le y \le h(t,x), \\ \Omega_2(y-h_0) + u_1(t,x), & h(t,x) \le y \le h_0; \end{cases} \tag{12}$$

$$v(t,x,y) = \begin{cases} -y u_{0x}, & 0 \le y \le h(t,x), \\ (h-y)u_{1x} - h u_{0x}, & h(t,x) \le y \le h_0. \end{cases} \tag{13}$$

For the case presented in Fig. 1a (Fig. 1b), $\Omega_1 = \omega_1$ and $\Omega_2 = \omega_2$ ($\Omega_1 = \omega_2$ and $\Omega_2 = \omega_1$). The configurations shown in Fig. 1a (Fig. 1b) occur for positive (negative) vorticities ω_i. We note that it is sufficient to examine one of these cases because, by virtue of symmetry about the central line of the channel, they are reduced to each other by an appropriate change of variables.

Using the solution representation (12) and the continuity condition for the velocity at the interface between the flows with constant vorticities Ω_1 and Ω_2, we express u_1 in terms of u_0 and h:

$$u_1(t,x) = (\Omega_1 - \Omega_2)h(t,x) + u_0(t,x) + \Omega_2 h_0. \tag{14}$$

By virtue of (13) and the condition $v(t,x,h_0) = 0$, we have the equation $(h_0 - h)u_{1x} + h u_{0x} = 0$, and integrating it with allowance for (14), we obtain

$$u_0 = \frac{\Omega_1 - \Omega_2}{2h_0}\, h^2 - (\Omega_1 - \Omega_2)h + a(t). \tag{15}$$

The fluid discharge in the channel is equal to zero:

$$Q = \int_0^{h_0} u\, dy = a(t)h_0 + \frac{\Omega_2 h_0^2}{2} = 0;$$

therefore, $a(t) \equiv -\Omega_2 h_0/2$. To construct the solution, it is necessary to satisfy the kinematic condition $h_t + u(t,x,h)h_x = v(t,x,h)$ at the interface $y = h(t,x)$ between the flows with different constant vorticities. In view of (12) and (13), these condition becomes

$$h_t + (\Omega_1 h + u_0)h_x + h u_{0x} = 0.$$

With the use of (15), the last equation is reduced to the scalar conservation law

$$\frac{\partial h}{\partial t} + \frac{\partial \varphi}{\partial x} = 0, \qquad \varphi(h) = \frac{\Omega_1 - \Omega_2}{2h_0} h^3 + \Big(\Omega_2 - \frac{\Omega_1}{2}\Big)h^2 - \frac{\Omega_2 h_0}{2}\, h. \tag{16}$$

Thus, constructing the solution of the problem of motion of a two-layer fluid with piecewise constant vorticity reduces to integrating Eq. (16). At the initial time ($t = 0$), we have $h = h_0$ at $x > 0$ and $h = 0$ at $x < 0$ or $h = 0$ at $x > 0$ and $h = h_0$ at $x < 0$, depending on the vorticities ω_1 and ω_2.

The condition of strong nonlinearity of the characteristic of Eq. (16) implies convexity of the function $\varphi(h)$ for all $h \in (0, h_0)$. We introduce the notation $\alpha_0 = \Omega_1/\Omega_2$. Then, the condition $\varphi''(h) \ne 0$ becomes

$$2^{-1} < \alpha_0 < 2. \tag{17}$$

Inequality (17) implies that the parameter $\alpha = \omega_1/\omega_2$ belongs to the interval $(1/2, 2)$.

3. Simple Wave of Flow Interaction. If inequality (17) is satisfied, the examined problem has a continuous solution specified by a simple wave. Because Eq. (16) and initial data (5) are invariant under uniform extension of the variables x and t, the solution is sought in the form of a self-similar simple wave $h = h(k)$ ($k = x/t$). We obtain the equation $(\varphi'(h) - k)h'(k) = 0$, by virtue of which

$$k = \varphi'(h) = \frac{3(\Omega_1 - \Omega_2)}{2h_0}\, h^2 + (2\Omega_2 - \Omega_1)h - \frac{\Omega_2 h_0}{2} \tag{18}$$

for $h'(k) \ne 0$. Solution of the characteristic equation (9) gives the same result.

From formulas (11) it follows that $u_1^r - u_0^l = u_1^l - u_0^r$; therefore, one of the two inequalities is valid: $u_1^r > u_0^l$ (see Fig. 1a) or $u_0^r > u_1^l$ (see Fig. 1b). With the use of ω_i, these inequalities are written as $\omega_1 + \omega_2 > 0$ or $\omega_1 + \omega_2 < 0$. In the region of flow interaction, the characteristic root k varies from u_0^l to u_1^r or from u_1^l to u_0^r. Because the flow is symmetric about the central line of the channel, it is sufficient to consider the case $\omega_1 + \omega_2 > 0$; in this case, the initial data for Eq. (16) are given by

$$h(0, x) = \begin{cases} 0, & x < 0, \\ h_0, & x > 0. \end{cases} \tag{19}$$

Differentiating (18), we obtain the following expressions for the first and second derivatives of the function $h(k)$:

$$h'(k) = \Big(\frac{3(\Omega_1 - \Omega_2)}{h_0}\, h(k) + 2\Omega_2 - \Omega_1\Big)^{-1}, \qquad h''(k) = -\frac{3(\Omega_1 - \Omega_2)}{h_0}\, (h'(k))^3. \tag{20}$$

From (20) it follows that for $\Omega_i > 0$, the inequality $h'(k) > 0$ holds; in the case $0 < \Omega_2 < \Omega_1$, the function $h(k)$ is convex upward, and in the case $0 < \Omega_1 < \Omega_2$, it is convex downward.

Let $0 < \omega_2 < \omega_1$, $\omega_1/\omega_2 = \alpha < 2$, $\Omega_1 = \omega_1$, and $\Omega_2 = \omega_2$. Different cases are analyzed similarly. Solution of Eq. (16) yields the line $y = h(k)$ [$k = x/t \in (u_0^l, u_1^r)$] that separates the flows with different vorticities:

$$h(k) = -\frac{(2\omega_2 - \omega_1)h_0}{3(\omega_1 - \omega_2)} + \sqrt{\frac{(2\omega_2 - \omega_1)^2 h_0^2}{9(\omega_1 - \omega_2)^2} + \frac{h_0(2k + \omega_2 h_0)}{3(\omega_1 - \omega_2)}} \tag{21}$$

[we choose the branch of the solution with the plus sign ahead of the square root since $\omega_1 > \omega_2$ and $h'(k) > 0$]. Substituting the function $h(x/t)$ into formulas (15) and (14) and using the solution representation (12), (13), we find the velocity field (u, v) in the flow interaction wave. The pressure on the upper boundary of the channel p^* is found by integrating any of Eqs. (3). It should be noted that the boundaries of the region of flow interaction move at the characteristic velocity. Indeed, as $h \to 0$ ($h \to h_0$), the characteristic root $k \to u_0^l = x_2'(t)$ [$k \to u_1^r = x_1'(t)$].

For a qualitative analysis of the solution, we find the values of the functions $h(k)$, $u_{1,2}(k)$, and $\bar u(k)$ and their first derivatives at the points A and C (for $k = u_0^l,\, u_1^r$):

$$h(u_0^l) = 0, \quad h(u_1^r) = h_0, \quad u_0(u_0^l) = u_0^l, \quad u_0(u_1^r) = u_0^r, \quad u_1(u_0^l) = u_1^l, \quad u_1(u_1^r) = u_1^r.$$

The first formula of (20) implies that

$$h'(u_0^l) = -(\omega_1 - 2\omega_2)^{-1}, \qquad h'(u_1^r) = (2\omega_1 - \omega_2)^{-1}.$$

Taking into account the relationship between the derivatives

$$u_0' = ((\omega_1 - \omega_2)h/h_0 - \omega_1 + \omega_2)h', \quad u_1' = (\omega_1 - \omega_2)h' + u_0', \quad \bar u' = \omega_1 h' + u_0',$$

we obtain the values of the functions $u_{1,2}'(k)$ and $\bar u(k)$ at the points A and C:

$$\begin{aligned} &u_0'(u_0^l) = \frac{\alpha - 1}{\alpha - 2}, \qquad u_1'(u_0^l) = 0, \qquad \bar u'(u_0^l) = \frac{1}{2 - \alpha}, \\ &u_0'(u_1^r) = 0, \qquad u_1'(u_1^r) = \frac{\alpha - 1}{2\alpha - 1}, \qquad \bar u'(u_1^r) = \frac{\alpha}{2\alpha - 1}. \end{aligned} \tag{22}$$

4. Particle Trajectories. The integral curves $x = x(t)$ and $y = y(t)$ of the systems of differential equations $x'(t) = u(k, y)$ and $y'(t) = v(t, x, y) = t^{-1}V(k, y)$ ($k = x/t$) specify the particle trajectories. Let us pass to the plane of the variables (k, y) and change the variable $s = \ln(t/t_0)$. Then, the equations for the trajectories become

$$\frac{dk}{ds} = u(k, y) - k, \qquad \frac{dy}{ds} = V(k, y). \tag{23}$$

Let us consider the region $ACF = \{(k, y)\colon\ u_0^l < k < u_1^r,\ 0 < y < h(k)\}$ (see Fig. 1), in which, according to the solution representation (12), (13), we have

$$u(k, y) = \omega_1 y + u_0(k), \qquad V(k, y) = -y u_0'(k).$$

The singular points of system (23) are the points $A = (u_0^l, 0)$ and $C = (u_1^r, h_0)$. Using (22), we linearize Eqs. (23) in the neighborhood of the point A:

$$\frac{dk}{ds} = \omega_1 y - \frac{k - u_0^l}{2 - \alpha}, \qquad \frac{dy}{ds} = \frac{\alpha - 1}{2 - \alpha}\, y.$$

The eigenvalues of the coefficient matrix

$$\lambda_1 = -(2-\alpha)^{-1} < 0, \qquad \lambda_2 = (2-\alpha)(\alpha-1) > 0$$

are real and have opposite signs ($1 < \alpha < 2$); therefore, the singular point A is a saddle. Equations (23) linearized in the neighborhood of the point C are written as

$$\frac{dk}{ds} = \omega_1(y-h_0) - (k-u_1^r), \qquad \frac{dy}{ds} = -h_0 u_0''(u_1^r)(k-u_1^r).$$

Setting $k = u_1^r$ in the formula

$$u_0''(k) = \frac{\omega_1-\omega_2}{h_0}\,(h'(k))^2 + \Big(\frac{\omega_1-\omega_2}{h_0}\,h(k) - \omega_1 + \omega_2\Big)h''(k)$$

and taking into account (20) and (21), we find $u_0''(u_1^r)$:

$$u_0''(u_1^r) = \frac{\alpha(\alpha-1)}{\omega_1 h_0(2\alpha-1)^2}.$$

Calculation of the characteristic roots of the matrix of the right side of system (23) linearized in the neighborhood of the point C yields

$$\lambda_{1,2} = -\frac{1}{2}\Big(1 \mp \frac{1}{2\alpha-1}\Big) < 0.$$

Therefore, the singular point C is a node.

Let us consider the region $ABC = \{(k,y)\colon\ u_0^l < k < u_1^r,\ h(k) < y < h_0\}$ (see Fig. 1), in which the functions $u-k$ and V have the form

$$u - k = u_1(k) + \omega_2(y-h_0) - k, \qquad V = -(y-h(k))u_1'(k) - h(k)u_0'(k)$$

and vanish at the points $A = (u_0^l, 0)$ and $C = (u_1^r, h_0)$. Linearization of system (23) in the neighborhood of the singular point A yields

$$\frac{dk}{ds} = -(k-u_0^l) + \omega_2 y, \qquad \frac{dy}{ds} = \frac{\alpha-1}{\omega_2(\alpha-2)^2}\,(k-u_0^l).$$

Calculation of the characteristic roots of the coefficient matrix of the right side of the linearized system shows that the roots

$$\lambda_{1,2} = -\frac{1}{2}\Big(1 \mp \frac{\alpha}{2-\alpha}\Big)$$

are real and have opposite signs. Therefore, the point A is a saddle. Linearization Eqs. (23) in the neighborhood of the singular point C yields

$$\frac{dk}{ds} = \frac{\alpha}{1-2\alpha}\,(k-u_1^r) + \omega_2(y-h_0), \qquad \frac{dy}{ds} = \frac{\alpha-1}{1-2\alpha}\,(y-h_0).$$

The eigenvalues of the coefficient matrix

$$\lambda_{1,2} = -\frac{1}{2}\Big(1 \mp \frac{1}{2\alpha-1}\Big) < 0$$

are real and have the same sign. Therefore, the point C is a node.

The results of numerical integration of Eqs. (23) illustrate the established nature of the singularities at the points A and C (Fig. 2a). In the case $0 < \omega_1 < \omega_2$, $1/2 < \alpha < 1$, the points $A = (u_0^l, 0)$ and $C = (u_1^r, h_0)$ are also singular for Eqs. (23) describing particle trajectories. In this case, the point A is a node, and the point C is a saddle (Fig. 2b). Along the interface of the flows with different vorticities, jet flow is formed which has a substantially two-dimensional nature.

5. Discretization of Equations (8) and Differential Conservation Laws. The equations of shear fluid flow in a channel are integrodifferential, and, hence, the methods developed for the numerical calculation of differential systems of conservation laws cannot be applied to them. Following [7], we divide the segment $[0,1]$ into N intervals ($0 = \lambda_0 < \lambda_1 < \ldots < \lambda_{N-1} < \lambda_N = 1$) and introduce the notation

$$y_i = \Phi(t,x,\lambda_i), \qquad \eta_i = y_i - y_{i-1}, \qquad u_i = u(t,x,\lambda_i), \qquad \omega_i = (u_i-u_{i-1})/\eta_i, \qquad u_{ci} = (u_i+u_{i-1})/2.$$

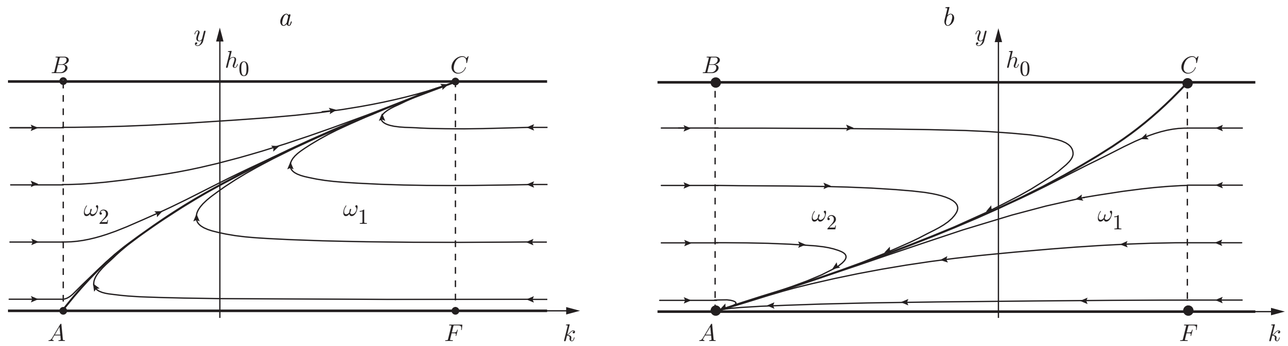

Fig. 2. Particle trajectories in a self-similar simple wave: (a) $0 < \omega_2 < \omega_1$; (b) $0 < \omega_1 < \omega_2$.

Integration of system (8) with respect to λ yields

$$\Big(\int\limits_{\lambda_{i-1}}^{\lambda_i} H\,d\lambda\Big)_t + \Big(\int\limits_{\lambda_{i-1}}^{\lambda_i} uH\,d\lambda\Big)_x = 0, \qquad (u_i - u_{i-1})_t + ((u_i^2 - u_{i-1}^2)/2)_x = 0,$$

$$u_N = -\Big(\sum_{i=1}^{N} \int\limits_{\lambda_{i-1}}^{\lambda_i} H\,d\lambda\Big)^{-1} \sum_{i=1}^{N} \int\limits_{\lambda_{i-1}}^{\lambda_i} wH\,d\lambda.$$

Taking into account that $H\,d\lambda = dy$ and applying the piecewise linear approximation of the velocity profile with depth

$$u = \omega_i(y - y_{i-1}) + u_{i-1}, \qquad y \in [y_{i-1}, y_i],$$

we obtain the following system of $2N$ differential equations for the quantities $(\eta_1, \ldots, \eta_N, \omega_1\eta_1, \ldots, \omega_N\eta_N)$:

$$\frac{\partial \eta_i}{\partial t} + \frac{\partial}{\partial x}\,(u_{ci}\eta_i) = 0, \qquad \frac{\partial}{\partial t}\,(\omega_i\eta_i) + \frac{\partial}{\partial x}\,(u_{ci}\omega_i\eta_i) = 0 \qquad (i = 1, \ldots, N), \tag{24}$$

$$u_{ci} = u_0 + \frac{\omega_i\eta_i}{2} + \sum_{j=1}^{i-1} \omega_j\eta_j, \qquad u_0 = -\Big(\sum_{i=1}^{N} \eta_i\Big)^{-1} \sum_{i=1}^{N} \eta_i\Big(\frac{\omega_i\eta_i}{2} + \sum_{j=1}^{i-1} \omega_j\eta_j\Big).$$

Equations (24) obtained by discretization of the integrodifferential conservation laws (8), describe the multilayer motion of an ideal fluid in a channel with a piecewise linear velocity profile along the depth in each of the N layers. A consequence of the system is the equation $(\eta_1 + \ldots + \eta_N)_t = 0$. If $\eta_1 + \ldots + \eta_N = h_0 = \text{const}$ at $t = 0$, the channel depth is equal to h_0 at all times. A numerical solution of system (24) can be obtained using methods developed to solve hyperbolic equations [8].

It should be noted that Eq. (16), which is used to describe two-layer fluid flow, is a particular case of system (24). Indeed, let us consider the case $N = 2$ with the initial data

$$\eta_i(0, x) = \begin{cases} 2^{-1}(1 + (-1)^i)h_0, & x < 0, \\ 2^{-1}(1 + (-1)^{i+1})h_0, & x > 0, \end{cases}$$

$$\omega_i\eta_i\Big|_{t=0} = \eta_i(0, x)\Omega_i \qquad (\Omega_i = \text{const}, \quad h_0 = \text{const}, \quad i = 1, 2).$$

Then, by virtue of (24), the vorticity in the layers retains constant values $\omega_i = \Omega_i$ during the interaction of the flows; therefore, the third and fourth equations for $\omega_1\eta_1$ and $\omega_2\eta_2$ coincide with the first two equations for η_1 and η_2. In view of the equality $\eta_1 + \eta_2 = h_0$ and the notation $\eta_1 = h$, the first two equations (24) reduce to the conservation law (16).

6. Discontinuity Solution. Solution of the problem of the interaction of shear flows with piecewise constant vorticities ω_i $(i = 1, 2)$ in a channel reduces to integrating the scalar conservation law (16), in which one

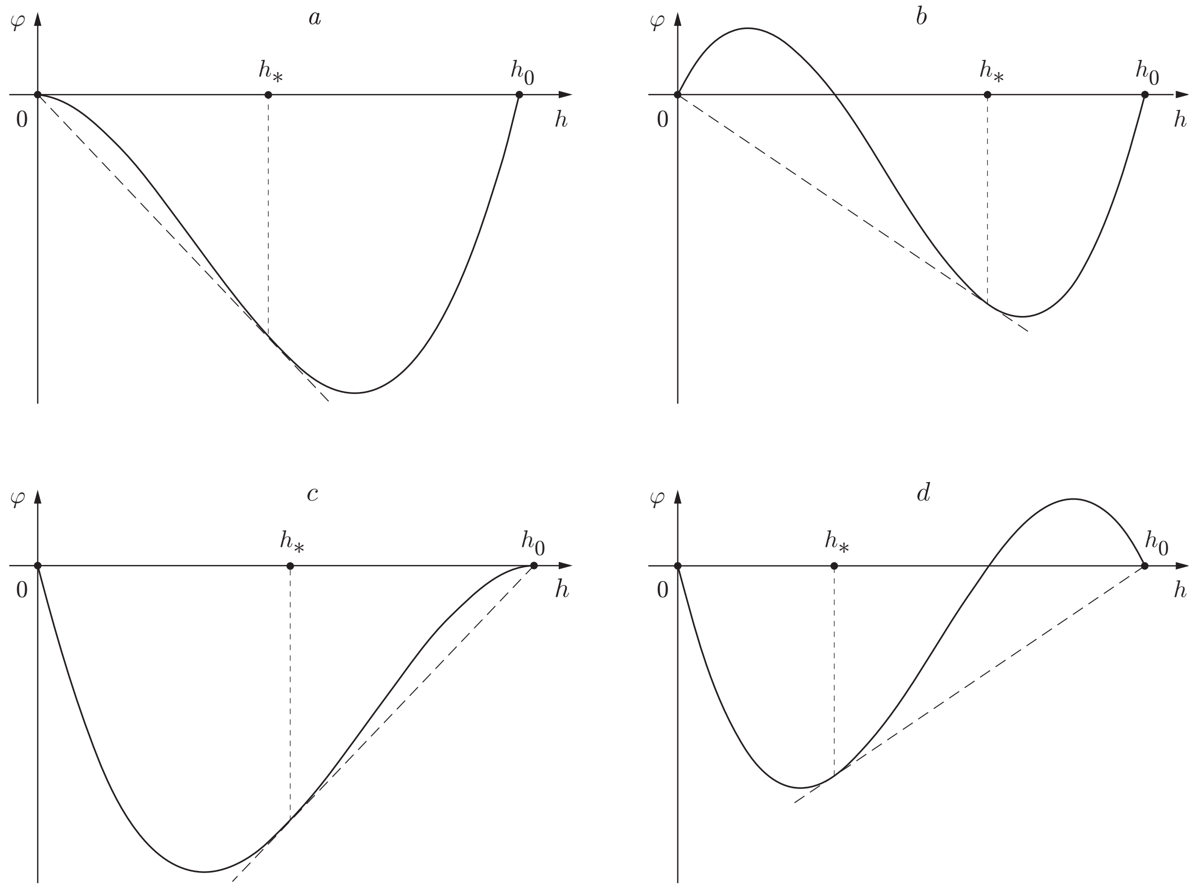

Fig. 3. Constructing a convex extension for the conservation law (16) in the following cases: (a) $0 < \Omega_2 < \Omega_1$ and $\Omega_1/\Omega_2 > 2$; (b) $\Omega_2 < 0 < \Omega_1$ and $|\Omega_2| < \Omega_1$; (c) $0 < \Omega_1 < \Omega_2$ and $\Omega_1/\Omega_2 < 1/2$; (d) $\Omega_1 < 0 < \Omega_2$ and $|\Omega_1| < \Omega_2$.

needs to set $\Omega_1 = \omega_1$ and $\Omega_2 = \omega_2$ for $\omega_1 + \omega_2 > 0$ or $\Omega_1 = \omega_2$ and $\Omega_2 = \omega_1$ for $\omega_1 + \omega_2 < 0$. If the function $\varphi(h)$ is not convex, Eq. (16) does not have a continuous solution that links the quantities $h = h^-$ and $h = h^+$ ($h^- = 0$, $h^+ = h_0$ or $h^- = h_0$, $h^+ = 0$).

We obtain a solution of Eq. (16) with initial data (19) in the form of a combination of a self-similar simple wave and a shock. According to the general theory of hyperbolic equations [9, 10], to obtain a steady-state discontinuous solution of the nonconvex conservation law, it is necessary to construct a convex extension taking into account the stability conditions for the discontinuity [11]:

$$\begin{aligned} &\frac{\varphi(h) - \varphi(h^-)}{h - h^-} \geq D \qquad (h^- < h < h^+), \\ &\frac{\varphi(h) - \varphi(h^+)}{h - h^+} \leq D \qquad (h^+ < h < h^-). \end{aligned} \tag{25}$$

As noted above, it is sufficient to consider the case $\Omega_1 + \Omega_2 > 0$, which corresponds to the initial data $h = h^- = 0$ for $x < 0$ and $h = h^+ = h_0$ for $x > 0$. For this, we use the first inequality (25) and construct the lower convex extension (Fig. 3) by replacing the segment $(0, h_*)$ [or (h_*, h_0)] of the curve $\varphi = \varphi(h)$ by a straight-line segment (dashed line). The slope of the dashed straight line (Fig. 3) is equal to the velocity of propagation of the discontinuity D, and the length of the segment on which the curve $\varphi = \varphi(h)$ is replaced by the straight-line segment is equal to the discontinuity amplitude h_*.

The quantities D and h_* are determined by solving the system of equations $\varphi(h_*) = Dh_*$ and $\varphi'(h_*) = D$ (Fig. 3a and b):

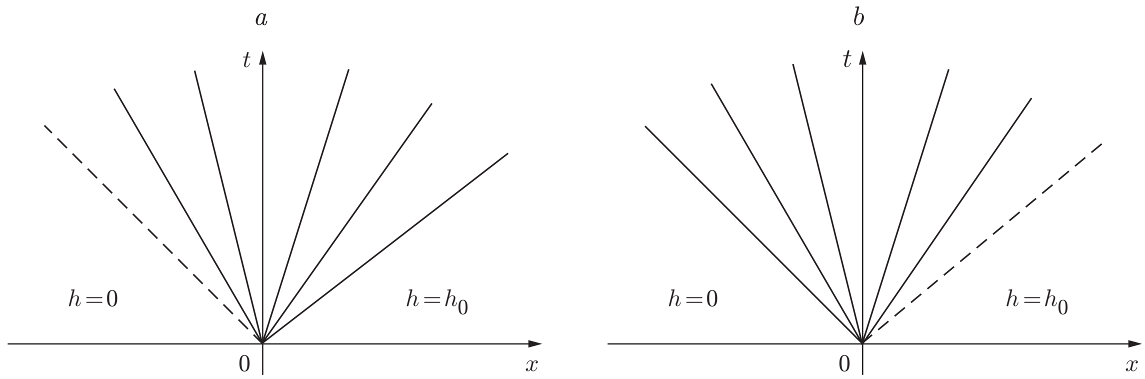

Fig. 4. Possible wave configurations: a steady-state discontinuity (dashed line) and a simple centered wave adjacent to the discontinuity front (solid lines): (a) $0 < \Omega_2 < \Omega_1$ ($\Omega_1/\Omega_2 > 2$) or $\Omega_2 < 0 < \Omega_1$ ($|\Omega_2| < \Omega_1$); (b) $0 < \Omega_1 < \Omega_2$ ($\Omega_1/\Omega_2 < 1/2$) or $\Omega_1 < 0 < \Omega_2$ ($|\Omega_1| < \Omega_2$).

$$D = \frac{\alpha_0^2 \Omega_2 h_0}{8(1-\alpha_0)}, \qquad h_* = \frac{(\alpha_0 - 2)h_0}{2(\alpha_0 - 1)} \qquad \Big(\alpha_0 = \frac{\Omega_1}{\Omega_2}\Big). \tag{26}$$

In the cases presented in Fig. 3c and d, the discontinuity velocity and amplitude are found similarly from the system $\varphi(h_*) = D(h_* - h_0)$, $\varphi'(h_*) = D$.

The picture of motion in the plane of (x, t) is shown in Fig. 4, where the dashed line shows a strong shock $x = Dt$, and the solid lines are a fan of characteristics of the simple wave, whose slope varies from D to u_1^r (Fig. 4a) or from u_0^l to D (Fig. 4b).

Let us consider two cases where the specified constant vorticities ω_i satisfy the inequalities.

Case 1: $0 < \omega_2 < \omega_1$ and $\omega_1/\omega_2 > 2$;

Case 2: $\omega_2 < 0 < \omega_1$ and $|\omega_2| < \omega_1$.

Cases 1 and 2 cover all qualitative features of the solution of the shear flow interaction problem for the model used. In this case, one needs to set $\Omega_1 = \omega_1$, $\Omega_2 = \omega_2$, and $\alpha_0 = \alpha = \omega_1/\omega_2$ and use the function $h(x)$, equal to zero for $x < 0$ and equal to h_0 for $x > 0$, as the initial data for Eq. (16).

Discontinuous Solution 1. The solution of Eq. (16) has the form

$$h(t, x) = \begin{cases} 0, & x < Dt, \\ \bar{h}(k), & Dt \le x \le \varphi'(h_0)t, \\ h_0, & x > \varphi'(h_0)t. \end{cases} \tag{27}$$

Here the discontinuity velocity D is given by the first formula in (26); the slope of the characteristic $\varphi'(h_0)$, along which the simple wave adjoins the specified shear flow, is equal to $u_1^r = \omega_1 h_0/2$; the function $\bar{h}(k)$ is defined by formula (21) with $k = x/t \in (D, \varphi'(h_0)]$. Using the known function $h(t, x) = \bar{h}(x/t)$ and formulas (12)–(15) and (3), it is easy to find the solution of the original problem (4) [the velocity field (u, v) and the pressure p]. In solutions of type 1, the ratio of the discontinuity amplitude to the channel depth $h_*/h_0 \in (0, 1/2)$ [the quantity h_* is defined by the second formula in (26); the minimum and maximum values of h_*/h_0 are reached for $\alpha \to 2$ and $\alpha \to \infty$, respectively].

In Euler coordinates, the relations at the discontinuity (10) become

$$[(u - D)^2 - (u_1 - D)^2] = 0, \qquad [(u - D)dy] = 0, \qquad [Q] = 0. \tag{28}$$

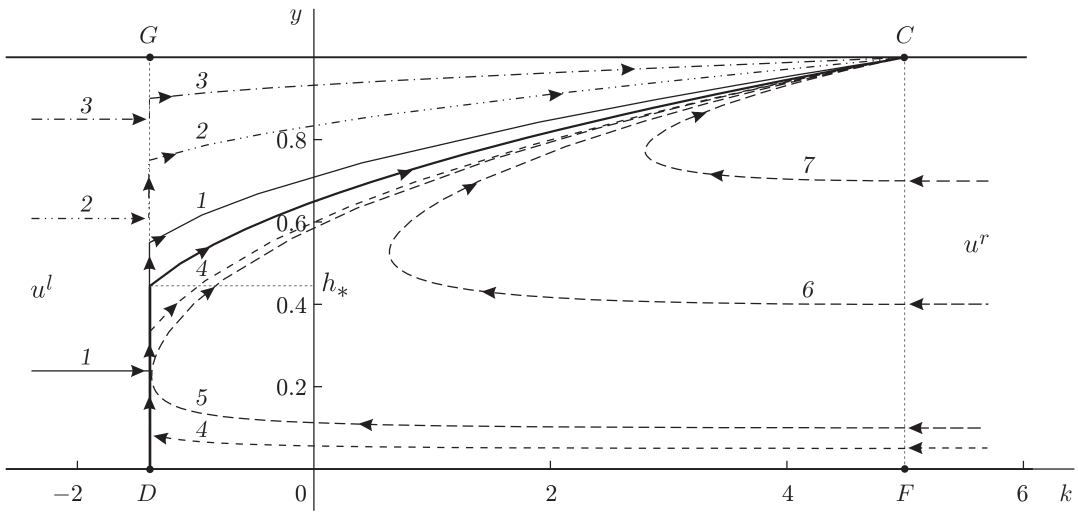

Fig. 5. Particle trajectories in the discontinuous solution 1 ($0 < \omega_2 < \omega_1$ and $\omega_1/\omega_2 > 2$).

Behind the discontinuity front, the horizontal velocity component $u(k, y)$ has the form

$$u(D+0, y^+) = u^+(y^+) = \begin{cases} \omega_1 y^+ + u_0^*, & 0 \le y^+ \le h_*, \\ \omega_2(y^+ - h_0) + u_1^*, & h_* \le y^+ \le h_0, \end{cases}$$

where

$$u_0^* = u_0(D+0) = -\frac{(3\alpha - 4)\alpha\omega_2 h_0}{8(\alpha - 1)}, \qquad u_1^* = u_1(D+0) = \frac{\alpha^2\omega_2 h_0}{8(\alpha - 1)} = -D.$$

Ahead of the discontinuity, $u(D-0, y) = u^l(y) = \omega_2 y + u_0^l$. Therefore, the second equation of (28) has the form

$$(u^+(y^+) - D)\, dy^+ = (u^l(y) - D)\, dy.$$

Solving this equation with allowance for $y^+(0) = h_*$, we obtain the function

$$y^+(y) = -\frac{\alpha - 2}{2}\, h_* + \sqrt{\frac{\alpha - 2}{2}\, h_* y + y^2 + \frac{\alpha^2 h_*^2}{4}} \qquad (0 \le y \le h_0), \tag{29}$$

which specifies the law of correspondence between the entry and exit points of the trajectories at the discontinuity front (in the integration, it was taken into account that $u^l(y) - D > 0$, $u^+(y^+) - D > 0$, $h_* \le y^+ \le h_0$).

The particle trajectories in the constructed discontinuous solution for $\omega_1 = 10$, $\omega_2 = 1$, $h_0 = 1$ are shown in Fig. 5. (The form of Fig. 5 is qualitatively the same for all values of ω_i that satisfy the inequalities in Case 1.) According to (29), the shear flow with vorticity ω_2 occupying the entire channel depth $0 \le y \le h_0$ ahead of the discontinuity $k = D - 0$ occupies only its part $h_* \le y \le h_0$ behind the discontinuity (trajectories 1–3 in Fig. 5). In the region $M = \{(k, y)\colon\ D < k < u_1^r,\ 0 \le y \le h(k)\}$, the particles perform return motion with respect to the wave, i. e., the quantity $u - k$ changes sign (trajectories 4–7 in Fig. 5). It should be noted that for $k = D$, the quantity $u^+(y) - D$ changes sign as y changes from 0 to h_*: $u^+(y_1) - D < 0$ if $y_1 \in [0, h_*/2)$ and $u^+(y_2) - D > 0$ if $y_2 \in (h_*/2, h_*]$. The solution of the equation

$$(u^+(y_1) - D)\, dy_1 = (u^+(y_2) - D)\, dy_2 \qquad [y_2(0) = h_*]$$

implies the law of correspondence between the entry and exit points of the trajectories $y_2 = h_* - y_1$ in the flow with vorticity ω_1 at the front $k = D$ follows. The constructed solution has a singularity at which the particles arrive at the discontinuity line from the region $k > D$ and, having changed the Euler coordinate y and the velocity vector at the discontinuity, return to the region $k > D$.

By the construction of the solution (see Fig. 5), the vorticity jump at the discontinuity is equal to zero: $[\omega] = [u_y] = 0$. From this relation and the condition $[(u - D)dy] = 0$, it follows that $[(u - D)^2 - (u_1 - D)^2] = 0$. The last equation of (28) is also valid for the solution obtained, and it is verified by direct calculations.

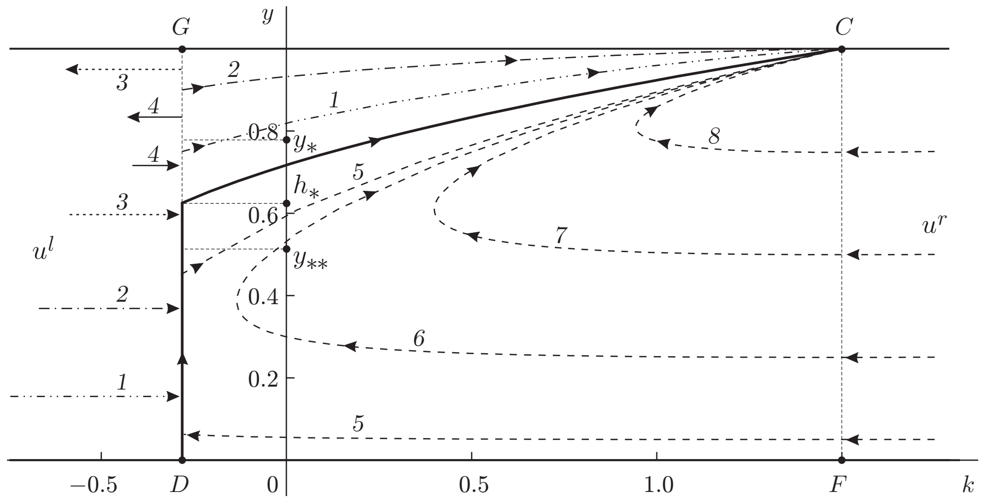

Fig. 6. Particle trajectories in the discontinuous solution 2 ($\omega_2 < 0 < \omega_1$ and $|\omega_2| < \omega_1$).

Let us show that the energy flux decreases in passing through a shock. For the smooth solutions of Eqs. (7), the energy conservation law for the fluid layer is satisfied:

$$\Big(\int\limits_0^1 u^2 H\,d\lambda\Big)_t + \Big(\int\limits_0^1 u^3 H\,d\lambda + 2p^* \int\limits_0^1 uH\,d\lambda\Big)_x = 0.$$

Because $Q = \int\limits_0^1 uH\,d\lambda = 0$, the total energy of the layer decreases if

$$\Big[\int\limits_0^{h_0} u^2 (u - D)\,dy\Big] \le 0.$$

Calculation of the integrals

$$e^l = \int\limits_0^{h_0} (u^l - D)(u^l)^2\,dy, \qquad e^+ = \int\limits_0^{h_0} (u^+ - D)(u^+)^2\,dy$$

for $\alpha > 2$ and $\omega_2 > 0$ yields the inequality $e^+ - e^l < 0$, which expresses the decrease in the energy flux at the discontinuity.

Discontinuous Solution 2. If the inequality $u^l - D \ge 0$ is satisfied for $\alpha < -2(1+\sqrt{2})$, the form of the solution corresponds qualitatively to the discontinuous solution 1. Let $-2(1+\sqrt{2}) < \alpha < -1$, i. e., the quantity $u^l - D$ changes sign. The line $y = h(t,x)$ separating the flows with vorticities ω_1 and ω_2 is defined by formula (27). It should be noted that $u^l - D > 0$ for $y \in [0, y_*)$ and $u^l - D < 0$ for $y \in (y_*, h_0]$, where

$$y_* = 8^{-1}(1-\alpha)^{-1}(2-\alpha)^2 h_0.$$

With the use of $y^+(0) = h_*$, the second relation in (28) reduces to the differential equation

$$(y - y_*)\,dy = (y^+ - 2y_*)\,dy^+.$$

Integration yields the law of correspondence between the entry and exit points of the trajectories at the discontinuity front:

$$y^+(y) = \frac{2-\alpha}{2}\,h_* - \sqrt{y^2 - \frac{2-\alpha}{2}\,h_* y + \frac{\alpha^2 h_*^2}{4}} \qquad \Big(0 \le y \le y_{**} = \frac{\alpha^2 h_0}{4(1-\alpha)}\Big). \tag{30}$$

Relation (30) is monotonic and $y^+(y_{**}) = h_0$.

The particle trajectories in the discontinuous solution 2 for $\omega_1 = 3$, $\omega_2 = -1$, and $h_0 = 1$ are shown in Fig. 6. The particles located in the layer $y \in [0, y_{**}]$ ahead of the discontinuity $k = D - 0$ occupy the layer $[h_*, h_0]$ (curves 1 and 2) behind the discontinuity $k = D + 0$. The particles occupying the layer $y \in (y_{**}, h_0]$ at $k = D - 0$ undergo a discontinuity (curves 3 and 4). At the discontinuity, the particles from the layer $y_1 \in [y_{**}, y_*)$ at $k = D - 0$ pass into the layer $y_2 \in (y_*, h_0]$ at $k = D - 0$. The law of correspondence between the trajectories is given by the formula $y_2(y_1) = h_0 + y_{**} - y_1$. We note that $u^l(y_1) - D > 0$ and $u^l(y_2) - D < 0$. In the region $M = \{(k, y)$: $D < k < u_1^r$, $0 \le y \le h(k)\}$, the particles perform rotational motion with respect to the wave (curves 5–8), and the trajectory entering the discontinuity $k = D$ undergo a jump (curve 5). The first and third relations of (28) are verified similarly to Case 1.

Conclusions. The problem of the interaction of ideal shear flows with piecewise constant vorticities was solved in the long-wave approximation. The existence of self-similar solutions describing all possible wave configurations that include a simple wave or a strong discontinuity and a simple wave was established. A system of differential equations for modeling the interaction of shear flows with arbitrary velocity profiles monotonic in depth was proposed.

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